

# RESTRICTING FOURIER TRANSFORMS OF MEASURES TO CURVES IN $\mathbb{R}^2$

M. BURAK ERDOĞAN   DANIEL M. OBERLIN

ABSTRACT. We establish estimates for restrictions to certain curves in  $\mathbb{R}^2$  of the Fourier transforms of some fractal measures.

## 1. INTRODUCTION

The starting point for this note was the following observation: if  $\mu$  is a compactly supported nonnegative Borel measure on  $\mathbb{R}^2$  which, for some  $\alpha > 3/2$ , is  $\alpha$ -dimensional in the sense that

$$(1.1) \quad \mu(B(y, r)) \lesssim r^\alpha$$

for  $y \in \mathbb{R}^2$  and  $r > 0$ , then

$$(1.2) \quad \int_0^\infty |\widehat{\mu}(t, t^2)|^2 dt < \infty.$$

The proof is easy: writing  $d\lambda$  for the measure given by  $dt$  on the curve  $(t, t^2)$ , we see that

$$(1.3) \quad \int_0^\infty |\widehat{\mu}(t, t^2)|^2 dt = \iiint e^{-2\pi i(t, t^2) \cdot (x-y)} d\mu(x) d\mu(y) dt = \iint \widehat{\lambda}(x-y) d\mu(x) d\mu(y) \lesssim \iint |x_2 - y_2|^{-1/2} d\mu(x) d\mu(y),$$

where we put  $x = (x_1, x_2)$  if  $x \in \mathbb{R}^2$  and the inequality comes from the van der Corput estimate  $|\widehat{\lambda}(x)| \lesssim |x_2|^{-1/2}$ . For fixed  $y$ , the compact support of  $\mu$  implies that

$$\int |x_2 - y_2|^{-1/2} d\mu(x) \lesssim \sum_{j=0}^\infty 2^{j/2} \mu(\{x : |x_2 - y_2| \leq 2^{-j}\}) \lesssim \sum_{j=0}^\infty 2^{j/2} 2^j 2^{-j\alpha}$$

since  $\{x : |x_2 - y_2| \leq 2^{-j}\}$  can be covered by  $\lesssim 2^j$  balls of radius  $2^{-j}$ . Clearly the last sum is finite if  $\alpha > 3/2$ , and then (1.3) is finite since  $\mu$  is a finite measure.

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The simplemindedness of this argument made it seem unlikely that the index  $3/2$  is best possible, and the search for that best index was the motivation for this work. Our results here are the following theorems:

**Theorem 1.1.** *Suppose  $\phi \in C^2([1, 2])$  satisfies the estimates*

$$(1.4) \quad \phi' \approx m, \phi'' \approx m$$

*for some  $m \geq 1$ , and let  $\gamma(t) = (t, \phi(t))$ . Suppose  $\mu$  is a nonnegative and compactly supported measure on  $\mathbb{R}^2$  which is  $\alpha$ -dimensional in the sense that (1.1) holds. Then, for  $\epsilon > 0$ ,*

$$(1.5) \quad \int_1^2 |\widehat{\mu}(R\gamma(t))|^2 dt \lesssim R^{-\alpha/2+\epsilon} m^{1-\alpha},$$

*when  $R \geq 2$ . Here the implied constant in (1.5) depends only on  $\alpha$ ,  $\epsilon$ , the implied constants in (1.1) and (1.4), and the diameter of the support of  $\mu$ .*

**Theorem 1.2.** *Suppose  $\mu$  is as in Theorem 1.1,  $p > 1$ , and*

- (i)  $-1 < \gamma < \alpha p - \alpha/2 - p$  if  $1 < \alpha < 2$ ,*
- (ii)  $-1 < \gamma < -1/2$  if  $1/2 < \alpha \leq 1$ ,*
- (iii)  $-1 < \gamma < \alpha - 1$  if  $0 < \alpha \leq 1/2$ .*

*Then*

$$(1.6) \quad \int_0^\infty |\widehat{\mu}(t, t^p)|^2 t^\gamma dt \leq C < \infty,$$

*where  $C$  depends only on  $p$ , the implied constant in (1.1), and the diameter of the support of  $\mu$ .*

**Theorem 1.3.** *If (1.6) holds for  $p > 1$  and  $\alpha \in (0, 2)$  with  $C$  as stated in Theorem 1.2, then*

- (i)  $-1 < \gamma \leq \alpha p - \alpha/2 - p$  if  $1 < \alpha < 2$ ,*
- (ii)  $-1 < \gamma \leq -1/2$  if  $1/2 < \alpha \leq 1$ ,*
- (iii)  $-1 < \gamma \leq \alpha - 1$  if  $0 < \alpha \leq 1/2$ .*

Here are some comments:

- (a) Theorem 1.1 is a generalization of Theorem 1 in [7], which was reproved with a simpler argument in [1]. As described in §2, the proof of Theorem 1.1 is just an adaptation of ideas from [7] and [1].
- (b) The examples which comprise the proof of Theorem 1.3 are similar in spirit to those in the proof of Proposition 3.2 in [7].
- (c) If  $\alpha_0$  is the infimum of the  $\alpha$ 's for which (1.1) implies (1.2) whenever  $\mu$  is compactly supported, it follows from Theorem 1.2 that  $\alpha_0 \leq 4/3$ . Then the proof of Theorem 1.3 and a uniform boundedness argument together imply that  $\alpha_0 = 4/3$ .
- (d) Analogs of Theorem 1.1 have been studied for hypersurfaces in  $\mathbb{R}^d$  and, particularly, for the sphere  $S^{d-1}$ . See, for example, [3], [4], [5], [6], [1], and [2].

The remainder of this note is organized as follows: the proof of Theorem 1.1 is in §2 and the proofs of Theorems 1.2 and 1.3 are in §3.

## 2. PROOF OF THEOREM 1.1

As mentioned above, the proof is an adaptation of ideas from [7] and [1]. Specifically, with  $\mu$  as in Theorem 1.1 and

$$\Gamma_R = \{R\gamma(t) : 1 \leq t \leq 2\}, \Gamma_{R,\delta} = \Gamma_R + B(0, R^\delta)$$

for  $R \geq 2$  and  $\delta > 0$ , we will modify an uncertainty principle argument from [7] to show that (1.5) follows from the estimate

$$(2.1) \quad \int_{\Gamma_{R,\delta}} |\widehat{\mu}(y)|^2 dy \lesssim R^{1-\alpha/2+2\delta} m^{2-\alpha}.$$

We will then adapt a bilinear argument from [1] to prove (2.1).

So, arguing as in [7], if  $\kappa \in C_c^\infty(\mathbb{R}^2)$  is equal to 1 on the support of  $\mu$ , then

$$(2.2) \quad \int_1^2 |\widehat{\mu}(R\gamma(t))|^2 dt = \int_1^2 \left| \int \widehat{\kappa}(R\gamma(t) - y) \widehat{\mu}(y) dy \right|^2 dt \lesssim \int \int_1^2 |\widehat{\kappa}(R\gamma(t) - y)| dt |\widehat{\mu}(y)|^2 dy.$$

If  $y = (y_1, y_2)$ , then

$$\begin{aligned} \int_1^2 |\widehat{\kappa}(R\gamma(t) - y)| dt &\lesssim \int_1^2 \frac{1}{(1 + |R\gamma(t) - y|)^{10}} dt \lesssim \\ &\frac{1}{(1 + \text{dist}(\Gamma_R, y))^8} \int_1^2 \frac{1}{(1 + |R\phi(t) - y_2|)^2} dt. \end{aligned}$$

Estimating the last integral using the hypothesized lower bound on  $\phi'$ , we see from (2.2) that

$$(2.3) \quad \int_1^2 |\widehat{\mu}(R\gamma(t))|^2 dt \lesssim \frac{1}{Rm} \int \frac{|\widehat{\mu}(y)|^2}{(1 + \text{dist}(\Gamma_R, y))^8} dy.$$

Now

$$\begin{aligned} \int \frac{|\widehat{\mu}(y)|^2}{(1 + \text{dist}(\Gamma_R, y))^8} dy &= \int_{\Gamma_{R,\epsilon/2}} + \sum_{j=2}^{\infty} \int_{\Gamma_{R,j\epsilon/2} \sim \Gamma_{R,(j-1)\epsilon/2}} \lesssim \\ &\int_{\Gamma_{R,\epsilon/2}} |\widehat{\mu}(y)|^2 dy + \sum_{j=2}^{\infty} R^{-8(j-1)\epsilon/2} \int_{\Gamma_{R,j\epsilon/2}} |\widehat{\mu}(y)|^2 dy. \end{aligned}$$

Thus (1.5) follows from (2.1) and (2.3).

Turning to the proof of (2.1), we note that by duality (and the fact that  $\mu$  is finite) it is enough to suppose that  $f$ , satisfying  $\|f\|_2 = 1$ , is supported on  $\Gamma_{R,\delta}$  and then to establish the estimate

$$(2.4) \quad \int |\widehat{f}(y)|^2 d\mu(y) \lesssim R^{1-\alpha/2+2\delta} m^{2-\alpha}.$$

The argument we will give differs from the proof of Theorem 3 in [1] only in certain technical details. But because those details are not always obvious, and for the convenience of any reader, we will give the complete proof.

For  $y \in \mathbb{R}^2$ , write  $y'$  for the point on the curve  $\Gamma_R$  which is closest to  $y$  (if there are multiple candidates for  $y'$ , choose the one with least first coordinate). Then  $y' = R\gamma(t')$  for some  $t' \in [1, 2]$ . For a dyadic interval  $I \subset [1, 2]$ , define

$$\Gamma_{R,\delta,I} = \{y \in \Gamma_{R,\delta} : t' \in I\}, \quad f_I = f \cdot \chi_{\Gamma_{R,\delta,I}}.$$

For dyadic intervals  $I, J \subset [1, 2]$ , we write  $I \sim J$  if  $I$  and  $J$  have the same length and are not adjacent but have adjacent parent intervals. The decomposition

$$(2.5) \quad [1, 2] \times [1, 2] = \bigcup_{n \geq 2} \left( \bigcup_{\substack{|I|=|J|=2^{-n} \\ I \sim J}} (I \times J) \right)$$

leads to

$$(2.6) \quad \int |\widehat{f}(y)|^2 d\mu(y) \leq \sum_{n \geq 2} \sum_{\substack{|I|=|J|=2^{-n} \\ I \sim J}} \int |\widehat{f}_I(y) \widehat{f}_J(y)| d\mu(y).$$

Truncating (2.5) and (2.6) gives

$$(2.7) \quad \int |\widehat{f}(y)|^2 d\mu(y) \leq \sum_{4 \leq 2^n \leq R^{1/2}} \sum_{\substack{|I|=|J|=2^{-n} \\ I \sim J}} \int |\widehat{f}_I(y) \widehat{f}_J(y)| d\mu(y) + \sum_{I \in \mathcal{I}} \int |\widehat{f}_I(y)|^2 d\mu(y),$$

where  $\mathcal{I}$  is a finitely overlapping set of dyadic intervals  $I$  with  $|I| \approx R^{-1/2}$ .

To estimate the integrals on the right hand side of (2.7), we begin with two geometric observations. The first of these is that if  $I \subset [1, 2]$  is an interval with length  $\ell$ , then

$$\Gamma_{R,I} \doteq \{R(t, \phi(t)) : t \in I\}$$

is contained in a rectangle  $D$  with side lengths  $\lesssim R\ell m, R\ell^2$ , which we will abbreviate by saying that  $D$  is a  $(R\ell m) \times (R\ell^2)$  rectangle. (To see this, note that the sine of the angle between vectors  $(1, M)$  and  $(1, M + \kappa)$  is

$$\frac{\kappa}{\sqrt{1+M^2}\sqrt{1+(M+\kappa)^2}},$$

it follows from (1.4) that the angle between tangent vectors at the beginning and ending points of the curve  $\Gamma_{R,I}$  is  $\lesssim \ell/m$ . Since the distance between these two points is  $\lesssim R\ell m$ , it is clear that  $\Gamma_{R,I}$  is contained in a rectangle  $D$  of the stated dimensions.) Secondly, we observe that if  $\ell \gtrsim R^{-1/2}$ , then an  $R^\delta$  neighborhood of an  $(R\ell m) \times (R\ell^2)$  rectangle is contained in an  $(R^{1+\delta}\ell m) \times (R^{1+\delta}\ell^2)$  rectangle. It follows that if  $I$  has length  $2^{-n} \gtrsim R^{-1/2}$ , then the

support of  $f_I$  is contained in a rectangle  $D$  with dimensions  $(R^{1+\delta}2^{-n}m) \times (R^{1+\delta}2^{-2n})$ .

The next lemma is part of Lemma 3.1 in [1] (the hypothesis  $1 \leq \alpha \leq 2$  there is not necessary for the conclusion of that lemma). To state it, we introduce some notation:  $\phi$  is a nonnegative Schwartz function such that  $\phi(x) = 1$  for  $x$  in the unit cube  $Q$ ,  $\phi(x) = 0$  if  $x \notin 2Q$ , and, for each  $M > 0$ ,

$$|\widehat{\phi}| \leq C_M \sum_{j=1}^{\infty} 2^{-Mj} \chi_{2^j Q}.$$

For a rectangle  $D \subset \mathbb{R}^2$ ,  $\phi_D$  will stand for  $\phi \circ b$ , where  $b$  is an affine mapping which takes  $D$  onto  $Q$ .

**Lemma 2.1.** *Suppose that  $\mu$  is a non-negative Borel measure on  $\mathbb{R}^2$  satisfying (1.1). Suppose  $D$  is a rectangle with dimensions  $R_2 \times R_1$ , where  $R_2 \gtrsim R_1$ , and let  $D_{dual}$  be the dual of  $D$  centered at the origin. Then, if  $\tilde{\mu}(E) = \mu(-E)$ ,*

$$(2.8) \quad (\tilde{\mu} * |\widehat{\phi_D}|)(y) \lesssim R_2^{2-\alpha}, \quad y \in \mathbb{R}^2$$

and, if  $K \gtrsim 1$ ,  $y_0 \in \mathbb{R}^2$ , then

$$(2.9) \quad \int_{K \cdot D_{dual}} (\tilde{\mu} * |\widehat{\phi_D}|)(y_0 + y) dy \lesssim K^\alpha R_2^{1-\alpha} R_1^{-1}.$$

Now if  $I \in \mathcal{I}$  and  $\text{supp} f_I \subset D$  as above, the identity  $\widehat{f_I} = \widehat{f_I} * \widehat{\phi_D}$  implies that

$$|\widehat{f_I}| \leq (|\widehat{f_I}|^2 * |\widehat{\phi_D}|)^{1/2} \|\widehat{\phi_D}\|_1^{1/2} \lesssim (|\widehat{f_I}|^2 * |\widehat{\phi_D}|)^{1/2}$$

and so

$$(2.10) \quad \begin{aligned} \int |\widehat{f_I}(y)|^2 d\mu(y) &\lesssim \int (|\widehat{f_I}|^2 * |\widehat{\phi_D}|)(y) d\mu(y) = \\ &\int |\widehat{f_I}(y)|^2 (\tilde{\mu} * |\widehat{\phi_D}|)(-y) dy \lesssim \|f_I\|_2^2 R^{1-\alpha/2+2\delta} m^{2-\alpha}, \end{aligned}$$

where the last inequality follows from (2.8) and the fact that  $D$  has dimensions  $(R^{1/2+\delta}m) \times R^\delta$  since  $2^{-n} \approx R^{-1/2}$ . Thus the estimate

$$(2.11) \quad \sum_{I \in \mathcal{I}} \int |\widehat{f_I}(y)|^2 d\mu(y) \lesssim R^{1-\alpha/2+2\delta} m^{2-\alpha} \sum_{I \in \mathcal{I}} \|f_I\|_2^2 \lesssim R^{1-\alpha/2+2\delta} m^{2-\alpha}$$

follows from  $\|f\|_2 = 1$  and the finite overlap of the intervals  $I \in \mathcal{I}$  (which implies finite overlap for the supports of the  $f_I, I \in \mathcal{I}$ ).

To bound the principal term of the right hand side of (2.7), fix  $n$  with  $4 \leq 2^n \leq R^{1/2}$  and a pair  $I, J$  of dyadic intervals with  $|I| = |J| = 2^{-n}$  and  $I \sim J$ . Since  $I \sim J$ , the support of  $f_I * f_J$  is contained in a rectangle  $D$  with dimensions  $(R^{1+\delta}2^{-n}m) \times (R^{1+\delta}2^{-2n})$ . For later reference, let  $v$  be a

unit vector in the direction of the longer side of  $D$ . As in (2.10),

$$(2.12) \quad \int |\widehat{f_I}(y)\widehat{f_J}(y)| d\mu(y) \lesssim \int (|\widehat{f_I}\widehat{f_J}| * |\widehat{\phi_D}|)(y) d\mu(y) = \\ \int |\widehat{f_I}(y)\widehat{f_J}(y)| (\tilde{\mu} * |\widehat{\phi_D}|)(-y) dy.$$

Now tile  $\mathbb{R}^2$  with rectangles  $P$  having exact dimensions  $C \times (C2^{-n}m^{-1})$  for some large  $C > 0$  to be chosen later and having shorter axis in the direction of  $v$ . Let  $\psi$  be a fixed nonnegative Schwartz function satisfying  $\psi(y) = 1$  if  $y \in Q$ ,  $\widehat{\psi}(x) = 0$  if  $x \notin Q$ , and

$$(2.13) \quad \psi \leq C_M \sum_{j=1}^{\infty} 2^{-Mj} \chi_{2^j Q}.$$

Since  $\sum_P \psi_P^3 \approx 1$ , it follows from (2.12) that if  $f_{I,P}$  is defined by

$$\widehat{f_{I,P}} = \psi_P \cdot \widehat{f_I}$$

then

$$(2.14) \quad \int |\widehat{f_I}(y)\widehat{f_J}(y)| d\mu(y) \lesssim \\ \sum_P \left( \int |\widehat{f_{I,P}}(y)\widehat{f_{J,P}}(y)|^2 dy \right)^{1/2} \left( \int |(\tilde{\mu} * |\widehat{\phi_D}|)(-y)\psi_P(y)|^2 dy \right)^{1/2}.$$

To estimate the first integral in this sum, we begin by noting that the support of  $f_{I,P}$  is contained in  $\text{supp}(f_I) + P_{\text{dual}}$ , where  $P_{\text{dual}}$  is a rectangle dual to  $P$  and centered at the origin. Let  $\tilde{I}$  be the interval with the same center as  $I$  but lengthened by  $2^{-n}/10$  and let  $\tilde{J}$  be defined similarly. Since  $I \sim J$ , it follows that  $\text{dist}(\tilde{I}, \tilde{J}) \geq 2^{-n}/2$ . Now the support of  $f_I$  is contained in  $\Gamma_{R,I} + B(0, R^\delta)$  and  $P_{\text{dual}}$  has dimensions  $(m2^n C^{-1}) \times C^{-1}$  with the longer direction at an angle  $\lesssim 2^{-n}/m$  to any of the tangents to the curve  $(t, \phi(t))$  for  $t \in \tilde{I}$  (or  $t \in \tilde{J}$ ). Recalling that  $2^n \lesssim R^{1/2}$ , one can check that, if  $C$  is large enough,

$$\text{supp}(f_{I,P}) \subset \Gamma_{R,\tilde{I}} + B(0, CR^\delta)$$

and, similarly,

$$\text{supp}(f_{J,P}) \subset \Gamma_{R,\tilde{J}} + B(0, CR^\delta).$$

The following lemma will be proved at the end of this section:

**Lemma 2.2.** *Suppose  $\phi$  satisfies the estimates*

$$0 < \phi' \leq m_1 \text{ and } \phi'' \geq m_2$$

*with  $m_1 \geq 1$  and*

$$(2.15) \quad m_1, m_2 \approx m.$$

Suppose that the closed intervals  $\tilde{I}, \tilde{J} \subset [1, 2]$  satisfy  $\text{dist}(\tilde{I}, \tilde{J}) \geq c2^{-n}$ . Then, for  $\delta > 0$  and  $x \in \mathbb{R}^2$ , there is the following estimate for the two-dimensional Lebesgue measure of the intersection of translates of tubular neighborhoods of  $\Gamma_{R, \tilde{I}}$  and  $\Gamma_{R, \tilde{J}}$ :

$$(2.16) \quad |x + \Gamma_{R, \tilde{I}} + B(0, CR^\delta) \cap \Gamma_{R, \tilde{J}} + B(0, CR^\delta)| \lesssim R^{2\delta} 2^n m.$$

The implicit constant in (2.16) depends only on the implicit constants in (2.15) and the positive constants  $c$  and  $C$ .

It follows from Lemma 2.2 that for  $x \in \mathbb{R}^2$  we have

$$(2.17) \quad |x + \text{supp}(f_{I,P}) \cap \text{supp}(f_{J,P})| \lesssim R^{2\delta} 2^n m.$$

Now

$$\int |\widehat{f_{I,P}}(y) \widehat{f_{J,P}}(y)|^2 dy = \int |\widetilde{f_{I,P}} * f_{J,P}(x)|^2 dx$$

and

$$\begin{aligned} |\widetilde{f_{I,P}} * f_{J,P}(x)| &\leq \int |f_{I,P}(w-x) f_{J,P}(w)| dw \leq \\ &|x + \text{supp}(f_{I,P}) \cap \text{supp}(f_{J,P})|^{1/2} (|\widetilde{f_{I,P}}|^2 * |f_{J,P}|^2(x))^{1/2}. \end{aligned}$$

Thus, by (2.17),

$$(2.18) \quad \left( \int |\widehat{f_{I,P}}(y) \widehat{f_{J,P}}(y)|^2 dy \right)^{1/2} \lesssim R^\delta 2^{n/2} m^{1/2} \left( \int |\widetilde{f_{I,P}}|^2 * |f_{J,P}|^2(x) dx \right)^{1/2} = R^\delta 2^{n/2} m^{1/2} \|f_{I,P}\|_2 \|f_{J,P}\|_2.$$

To estimate the second integral in the sum (2.14) we use (2.13) to observe that

$$\psi_P \lesssim \sum_{j=1}^{\infty} 2^{-Mj} \chi_{2^j P}.$$

Thus

$$\int (\tilde{\mu} * |\widehat{\phi_D}|)(-y) \psi_P(y) dy \lesssim \sum_{j=1}^{\infty} 2^{-Mj} \int_{2^j P} (\tilde{\mu} * |\widehat{\phi_D}|)(-y) dy.$$

Noting that  $2^j P \subset y_P + KD_{\text{dual}}$  for some  $K \lesssim R^{1+\delta} 2^{-2n+j}$  and some  $y_P \in \mathbb{R}^2$ , we apply (2.9) to obtain

$$\begin{aligned} &\int (\tilde{\mu} * |\widehat{\phi_D}|)(-y) \psi_P(y) dy \lesssim \\ &\sum_{j=1}^{\infty} 2^{-Mj} (R^{1+\delta} 2^{-2n+j})^\alpha (R^{1+\delta} 2^{-n} m)^{1-\alpha} (R^{1+\delta} 2^{-2n})^{-1} \lesssim 2^{-n(\alpha-1)} m^{1-\alpha}. \end{aligned}$$

Since

$$(\tilde{\mu} * |\widehat{\phi_D}|)(-y) \lesssim (R^{1+\delta} 2^{-n} m)^{2-\alpha}$$

by (2.8) and since  $\psi_P(y) \lesssim 1$ , it follows that

$$(2.19) \quad \left( \int ((\tilde{\mu} * |\widehat{\phi_D}|)(-y)\psi_P(y))^2 dy \right)^{1/2} \lesssim R^{1-\alpha/2+\delta(1-\alpha/2)} 2^{-n/2} m^{3/2-\alpha}.$$

Now (2.18) and (2.19) imply, by (2.14), that

$$\int |\widehat{f_I}(y)\widehat{f_J}(y)| d\mu(y) \lesssim R^{1-\alpha/2+\delta(2-\alpha/2)} m^{2-\alpha} \left( \sum_P \|f_{I,P}\|_2^2 \right)^{1/2} \left( \sum_P \|f_{J,P}\|_2^2 \right)^{1/2}.$$

Since

$$\sum_P \|\widehat{f_{I,P}}\|_2^2 = \int |\widehat{f_I}(y)|^2 \sum_P |\psi_P(y)|^2 dy,$$

it follows from  $\sum_P \psi_P^2 \lesssim 1$  that

$$\int |\widehat{f_I}(y)\widehat{f_J}(y)| d\mu(y) \lesssim R^{1-\alpha/2+\delta(2-\alpha/2)} m^{2-\alpha} \|f_I\|_2 \|f_J\|_2.$$

Thus

$$(2.20) \quad \sum_{\substack{|I|=|J|=2^{-n} \\ I \sim J}} \int |\widehat{f_I}(y)\widehat{f_J}(y)| d\mu(y) \lesssim R^{1-\alpha/2+\delta(2-\alpha/2)} m^{2-\alpha} \sum_{\substack{|I|=|J|=2^{-n} \\ I \sim J}} \|f_I\|_2 \|f_J\|_2 \lesssim R^{1-\alpha/2+\delta(2-\alpha/2)} m^{2-\alpha} \|f\|_2^2.$$

Now (2.4) follows from (2.7), (2.11), (2.20), and the fact that the first sum in (2.7) has  $\lesssim \log R$  terms.

Here is the proof of Lemma 2.2:

*Proof.* Fix  $t \in \tilde{I}$ ,  $s \in \tilde{J}$  such that

$$(2.21) \quad x + R(t, \phi(t)) + \overline{B(0, CR^\delta)} \cap R(s, \phi(s)) + \overline{B(0, CR^\delta)} \neq \emptyset$$

and such that  $t$  is minimal subject to (2.21). Without loss of generality, assume that  $t < s$ . Suppose that  $v$  and  $w$  satisfy

$$(2.22) \quad x + R(t+w, \phi(t+w)) + \overline{B(0, CR^\delta)} \cap R(s+v, \phi(s+v)) + \overline{B(0, CR^\delta)} \neq \emptyset.$$

We will begin by observing that

$$(2.23) \quad w \leq \frac{8C2^n R^{\delta-1} m_1}{c m_2}.$$

From (2.21) and (2.22) it follows that

$$(2.24) \quad |w-v|, |(\phi(s+v) - \phi(s)) - (\phi(t+w) - \phi(t))| \leq 4CR^{\delta-1}.$$

Now

$$(2.25) \quad (\phi(s+v) - \phi(s)) - (\phi(t+w) - \phi(t)) = \int_t^{t+w} (\phi'(u+s-t) - \phi'(u)) du + e$$



where the error term  $e$  satisfies  $|e| \leq 4CR^{\delta-1}m_1$  because of the first inequality in (2.24) and the bound on  $\phi'$ . Since  $s - t \geq c2^{-n}$ , the lower bound on  $\phi''$  shows that the integral in (2.25) exceeds  $wc2^{-n}m_2$ . Thus if

$$wc2^{-n}m_2 > 8CR^{\delta-1}m_1$$

(that is, if (2.23) fails) then, since  $m_1 \geq 1$ , (2.25) exceeds  $4CR^{\delta-1}$ , contradicting (2.24).

To see (2.16), define  $\tilde{t}$  by

$$\tilde{t} = t + \frac{8C2^n R^{\delta-1}m_1}{cm_2}$$

and note that by (2.23) the intersection in (2.16) is contained in a translate of

$$\{R(u, \phi(u)) : t \leq u \leq \tilde{t}\} + B(0, CR^\delta) \doteq \Gamma + B(0, CR^\delta).$$

Using  $\phi' \lesssim m$ , the length of the curve  $\Gamma$  is  $\lesssim 2^n R^\delta m$ . Thus  $\Gamma$  is contained in  $\lesssim 2^n m$  balls of radius  $R^\delta$ . This implies (2.16).  $\square$

### 3. PROOF OF THEOREMS 1.2 AND 1.3

*Proof of Theorem 1.2:* First suppose  $1 < \alpha < 2$ . Choose  $\epsilon > 0$  such that  $\gamma + 2\epsilon < \alpha(p - 1/2) - p$ . Then apply Theorem 1.1 with  $\phi(t) = R^{p-1}t^p$  and  $m = R^{p-1}$  to conclude that

$$\int_1^2 |\widehat{\mu}(Rt, (Rt)^p)|^2 dt \lesssim R^{-\alpha/2+\epsilon} R^{(p-1)(1-\alpha)}$$

and so

$$\int_R^{2R} |\widehat{\mu}(t, t^p)|^2 t^\gamma dt \lesssim R^{-\epsilon}.$$

Now (1.6) follows by taking  $R = 2^n$ .

To deal with the remaining cases we note that if  $d\nu$  is  $dt$  on the curve  $(t, R^{p-1}t^p)$ ,  $1 \leq t \leq 2$ , then there is the estimate  $|\widehat{\nu}(\xi)| \lesssim |\xi|^{-1/2}$ . It follows from Theorem 1 in [1] that

$$\int_1^2 |\widehat{\mu}(Rt, (Rt)^p)|^2 dt \lesssim R^{-\min(\alpha, 1/2)}.$$

This implies the conclusions of Theorem 1.3 in cases (ii) and (iii) exactly as in the preceding paragraph.

*Proof of Theorem 1.3:* We begin by observing that if the conclusion (1.6) of Theorem 1.2 holds for  $\alpha \in (0, 2)$  with  $C$  depending only on the size of the support of the nonnegative measure  $\mu$  and the implied constant in (1.1), then the same conclusion holds (with  $C$  replaced by  $16C$ ) for complex measures whose total variation measure  $|\mu|$  satisfies (1.1).

We consider first the case  $\alpha \in (1, 2)$ . Suppose  $R$  is large and positive. It is easy to check that the set

$$\{(t, t^p) : R \leq t \leq R + \sqrt{R}\}$$

is contained in a rectangle  $D$  with (approximate) dimensions  $1 \times R^{p-1/2}$ . Let  $v$  be a unit vector in the direction of the long axis of  $D$  and  $c_D$  be the center of  $D$ . Also, denote the dual of  $D$  centered at the origin by  $D_{\text{dual}}$ . Note that  $D_{\text{dual}}$  is a rectangle with dimensions  $1 \times R^{1/2-p}$  with short axis in the direction  $v$ . Fix a function  $\psi \in C_c^\infty$  supported in  $D_{\text{dual}}$  such that  $\widehat{\psi} \gtrsim R^{(p-1/2)(1-\alpha)}$  on  $D$  and  $\|\psi\|_\infty \lesssim R^{(p-1/2)(2-\alpha)}$ . Let  $T \approx R^{(p-1/2)(\alpha-1)}$  be a natural number and define  $\mu$  by

$$(3.1) \quad \mu(y) \doteq e^{2\pi i y \cdot c_D} \sum_{k=1}^T \psi(y - kT^{-1}v).$$

It is easy to check that  $|\mu|$  satisfies (1.1) independently of  $R$ . Also note that

$$|\widehat{\mu}(x)| \gtrsim R^{(p-1/2)(1-\alpha)} \chi_D(x) \left| \sum_{k=1}^T e^{-2\pi i \frac{k}{T} v \cdot (x - c_D)} \right|.$$

Now if

$$\left| \frac{1}{T} v \cdot (x - c_D) \right| \leq 1/4 \pmod{1},$$

then we have

$$\left| \sum_{k=1}^T e^{-2\pi i \frac{k}{T} v \cdot (x - c_D)} \right| \gtrsim T.$$

Therefore there are  $N \approx R^{p-1/2}/T \approx R^{(p-1/2)(2-\alpha)}$  subrectangles  $P_1, \dots, P_N$  of  $D$  with dimensions  $1 \times 1/4$  whose centers are in an arithmetic progression with distance  $T$  between the adjacent points such that

$$|\widehat{\mu}(x)| \gtrsim R^{(p-1/2)(1-\alpha)} T \sum_{k=1}^N \chi_{P_k}(x) \approx \sum_{k=1}^N \chi_{P_k}(x).$$

Using this we obtain

$$\begin{aligned} \int_R^{R+\sqrt{R}} |\widehat{\mu}(t, t^p)|^2 t^\gamma dt &\gtrsim R^\gamma \int_R^{R+\sqrt{R}} \sum_{k=1}^N \chi_{P_k}(t, t^p) dt \\ &\gtrsim R^\gamma \frac{N}{R^{p-1}} \approx R^{\gamma - \alpha p + \alpha/2 + p}. \end{aligned}$$

This implies that  $\gamma \leq \alpha p - \alpha/2 - p$  and so gives the conclusion (i) of Theorem 1.3.

The conclusion (ii) of Theorem 1.3 also follows from the examples just constructed: since the support of  $\mu$  above is contained in a ball of radius  $\approx 1$ , if  $|\mu|$  satisfies (1.1) for some  $\alpha > 1$ , then the same is certainly true for all  $\alpha \in (0, 1]$ . Taking  $\alpha = 1 + \delta$  for arbitrary  $\delta > 0$  gives  $\gamma \leq -1/2$ .

To conclude, suppose  $\alpha \in (0, 1/2)$  and  $R > 0$  is large. Let  $D$  be a rectangle with dimensions  $R \times R^p$  which contains

$$\{(t, t^p) : R \leq t \leq 2R\},$$

and let  $v$ ,  $C_D$ , and  $D_{\text{dual}}$  be as above. Note that now  $D_{\text{dual}}$  is a rectangle with dimensions  $R^{-1} \times R^{-p}$  with short axis in the direction  $v$ . Fix a function  $\psi \in C_c^\infty$  supported in  $D_{\text{dual}}$  and satisfying  $\widehat{\psi} \gtrsim R^{-\alpha}$  on  $D$  and  $\|\psi\|_\infty \lesssim R^{p+1-\alpha}$ . Fix a natural number  $T$  with  $T \approx R^\alpha$  and again define  $\mu$  by (3.1). As before,  $|\mu|$  satisfies (1.1) independently of  $R$  and there are  $N \approx R^p/T \approx R^{p-\alpha}$  disjoint subrectangles  $P_1, \dots, P_N$  of  $D$  of dimensions  $1 \times 1/4$  such that

$$|\widehat{\mu}(x)| \gtrsim R^{-\alpha} T \sum_{k=1}^N \chi_{P_k}(x) \approx \sum_{k=1}^N \chi_{P_k}(x).$$

As above, that leads to

$$\begin{aligned} \int_R^{2R} |\widehat{\mu}(t, t^p)|^2 t^\gamma dt &\gtrsim R^\gamma \int_R^{2R} \sum_{k=1}^N \chi_{P_k}(t, t^p) dt \\ &\gtrsim R^\gamma \frac{N}{R^{p-1}} \approx R^{\gamma+p-\alpha-(p-1)}. \end{aligned}$$

This gives the conclusion (iii) of Theorem 1.3.

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M. B. ERDOĞAN, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, IL 61801

*E-mail address:* berdogan@math.uiuc.edu

D. M. OBERLIN, DEPARTMENT OF MATHEMATICS, FLORIDA STATE UNIVERSITY, TALLAHASSEE, FL 32306

*E-mail address:* oberlin@math.fsu.edu